Immagine che contiene testo, lettera, carta, Carattere

Descrizione generata automaticamenteTheoretical definitions

a. Given sets A, B ⊆ N, we say that A is reducible to B, denoted A ≤\_m B, if there exists a total computable function f : N → N such that for all x ∈ N, x ∈ A if and only if f(x) ∈ B. The function f is called the reduction function.

b. Suppose A is not r.e. and A ≤\_m B. Let f be the computable reduction function such that x ∈ A iff f(x) ∈ B for all x ∈ N.

If B were r.e., its semi-characteristic function sc\_B would be computable. But then we could define sc\_A(x) = sc\_B(f(x)), which would be computable by composition, implying A is r.e. This contradicts the assumption that A is not r.e.

Therefore, if A is not r.e. and A ≤\_m B, then B cannot be r.e.

c. The statement does not hold in general. As a counterexample, consider A = N\{0} and B = {0}.

We have A ≤\_m B via the constant reduction function f(x) = 0 for all x ∈ N. However, the converse reduction does not exist because there is no computable function g : N → N such that 0 ∈ B iff g(0) ∈ A, since g(0) ∈ N\{0} for any total computable g.

The reducibility relation is transitive (if A ≤\_m B and B ≤\_m C then A ≤\_m C) but not symmetric in general. A sufficient condition for symmetry is requiring the reduction function to be bijective. If f reduces A to B and f is a computable bijection, then f^(-1) reduces B to A. However, bijective reductions are atypical.

In summary, the reducibility relation captures a notion of one set being "no harder" than another, but it does not imply equivalence unless the reduction is bijective. Sets need not be reducible to each other in both directions.

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Descrizione generata automaticamente

Given sets A, B ⊆ N, let us define the reduction A ≤\_m B as follows: A ≤\_m B holds if and only if there exists a total computable function f : N → N such that for all x ∈ N, x ∈ A iff f(x) ∈ B.

Now, suppose A ≤\_m B and A is not recursive. We want to show that B is also not recursive.

Assume, for the sake of contradiction, that B is recursive. Since A ≤\_m B, there exists a total computable function f : N → N such that x ∈ A iff f(x) ∈ B for all x ∈ N.

Let χ\_B be the characteristic function of B, which is computable since B is assumed to be recursive. Then we can define χ\_A : N → N as follows:

χ\_A(x) = χ\_B(f(x))

Observe that χ\_A is computable, since it is the composition of the computable functions χ\_B and f. Moreover, for any x ∈ N:

- If x ∈ A, then f(x) ∈ B, so χ\_B(f(x)) = 1, and thus χ\_A(x) = 1.

- If x ∉ A, then f(x) ∉ B, so χ\_B(f(x)) = 0, and thus χ\_A(x) = 0.

Therefore, χ\_A is the characteristic function of A, implying that A is recursive. However, this contradicts our assumption that A is not recursive.

Hence, our initial assumption that B is recursive must be false. We conclude that if A ≤\_m B and A is not recursive, then B is also not recursive.

The converse statement does not hold in general. That is, given A ≤\_m B, if B is not recursive, A may still be recursive. Here's a counterexample:

Let A = ∅ (the empty set) and B = K̅ (the complement of the halting set). We know that K̅ is not recursive (since K is not recursive).

Define the function f : N → N as f(x) = 0 for all x ∈ N.

Clearly, f is a total computable function. Moreover, for any x ∈ N:

x ∈ A iff x ∈ ∅ iff false iff 0 ∈ K̅ iff f(x) ∈ B

Thus, A ≤\_m B. However, A = ∅ is recursive (trivially), while B = K̅ is not recursive.

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Descrizione generata automaticamenteThis counterexample demonstrates that the converse of the original statement does not hold. The reduction A ≤\_m B, even coupled with the non-recursiveness of B, does not imply the non-recursiveness of A.

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Descrizione generata automaticamenteNon-computable functions

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Descrizione generata automaticamenteRecursiveness

Exercise 8.1 Solution:

The set A = {x ∈ N : |W\_x| ≥ 2} is not recursive. To prove this, first observe that A is saturated since A = {x | φ\_x ∈ A} where A = {f | |dom(f)| ≥ 2}.

Using Rice-Shapiro's theorem, we can deduce that both A and A̅ are not r.e.:

1. A is not r.e. No finite function can belong to A, since finite functions have finite domains (and thus domains of size < 2). However, A is non-empty, e.g. the identity function is in A. Therefore, by Rice-Shapiro's theorem, A is not r.e.

2. A̅ is also not r.e. The always undefined function ∅ is in A̅, but it admits the identity function as an extension which is not in A̅. Thus, by Rice-Shapiro's theorem, A̅ is not r.e.

Since both A and its complement are not r.e., we conclude that A is not recursive.

Exercise 8.2 Solution:

The set A = {x ∈ N : x ∈ W\_x ∩ E\_x} is not recursive, but it is r.e.

To show A is not recursive, we prove that K ≤\_m A. Consider the computable function:

g(x,y) = 1 if x ∈ K, and undefined otherwise

By the s-m-n theorem, there exists a total computable function s such that φ\_(s(x))(y) = g(x,y) for all x,y ∈ N.

The function s is a reduction of K to A:

- If x ∈ K, then φ\_(s(x))(y) = g(x,y) = 1 for all y. So s(x) ∈ W\_(s(x)) = E\_(s(x)) = N, and thus s(x) ∈ A.

- If x ∉ K, then φ\_(s(x))(y) = g(x,y) is undefined for all y. So W\_(s(x)) = E\_(s(x)) = ∅, hence s(x) ∉ A.

Therefore K ≤\_m A, implying A is not recursive.

However, A is r.e. since its semi-characteristic function

sc\_A(x) = μw.H(x,x,(w)\_1) ∨ S(x,(w)\_2,x,(w)\_1)

is computable. The μ-minimalization succeeds if either the machine x halts on input x, or if machine x produces output x for some input, both of which capture the definition of A.

In conclusion, the set A is r.e. but not recursive, so its complement A̅ is not r.e.

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Descrizione generata automaticamenteExercise 8.5 Solution:

Let A = {x ∈ N : ∃y, z ∈ N. z > 1 ∧ x = y^z}. We will show that A is not recursive by proving that K ≤\_m A.

Consider the computable function g(x,y) = 1 if x ∈ K, undefined otherwise. By the s-m-n theorem, there exists a total computable function s : N → N such that φ\_(s(x))(y) = g(x,y) for all x,y ∈ N.

We argue that s is a reduction function for K ≤\_m A:

- If x ∈ K, then φ\_(s(x))(y) = g(x,y) = 1 for all y ∈ N. So s(x) = s(x)^1, with s(x), 1 ∈ N and 1 > 1. Thus s(x) ∈ A.

- If x ∉ K, then φ\_(s(x))(y) = g(x,y) is undefined for all y ∈ N. So there cannot exist y, z ∈ N with z > 1 such that s(x) = y^z. Thus s(x) ∉ A.

Therefore, K ≤\_m A, implying A is not recursive.

The set A is r.e., since its semi-characteristic function

sc\_A(x) = μw.(S(x,(w)\_1,(w)\_3) ∧ (w)\_2 > 1 ∧ x = (w)\_1^(w)\_3)

is computable. Since A is r.e. but not recursive, its complement A̅ is not r.e. (otherwise both would be recursive). Thus A̅ is also not recursive.

Exercise 8.6 Solution:

Let A = {x ∈ N : φ\_x(y) = y for infinitely many y}. The set A is saturated, since A = {x : φ\_x ∈ A}, where A = {f : f(y) = y for infinitely many y}.

Applying Rice-Shapiro's theorem, we deduce that both A and A̅ are not r.e.:

1. A is not r.e. The identity function id ∈ A, but no finite subfunction θ ⊆ id can belong to A, since such θ would only satisfy θ(y) = y for finitely many y. Thus by Rice-Shapiro's theorem, A is not r.e.

2. A̅ is not r.e. The always undefined function ∅ ∈ A̅, but ∅ admits the identity function id as an extension, and id ∉ A̅. Thus by Rice-Shapiro's theorem, A̅ is not r.e.

Since both A and A̅ are not r.e., we conclude that A is not recursive. In summary, A is neither recursive nor recursively enumerable. The same holds for its complement A̅.

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Descrizione generata automaticamenteSmn-theorem

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Descrizione generata automaticamente

Second recursion theorem

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The Second Recursion Theorem states that given any total computable function h : N → N, there exists an e ∈ N such that φ\_e = φ\_(h(e)).

To prove the desired result, consider the function g : N^2 → N defined as follows:

g(x,y) = x

This function is clearly computable. Therefore, by the s-m-n theorem, there exists a total computable function s : N → N such that for all x,y ∈ N:

φ\_(s(x))(y) = g(x,y) = x

Now, by the Second Recursion Theorem applied to the function s, there exists an index e ∈ N such that:

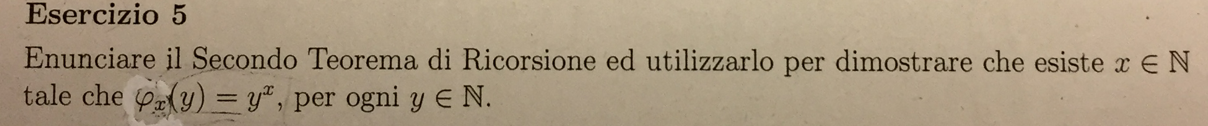
φ\_e = φ\_(s(e))

This means that for all y ∈ N:

φ\_e(y) = φ\_(s(e))(y) = g(e,y) = e

Hence, W\_e = {y | φ\_e(y) ↓} = {y | φ\_(s(e))(y) ↓} = {y | e ↓} = N

Therefore, |W\_e| = |N| = e, as desired.

In summary, by leveraging the power of the Second Recursion Theorem in conjunction with the s-m-n theorem, we have demonstrated the existence of a program index e such that the domain of the e-th computable function φ\_e has cardinality exactly equal to e itself. This construction relies on the ability to define a suitable computable function that, when combined with the recursion theorem, yields the desired property.

The Second Recursion Theorem states that for any total computable function h : N → N, there exists an e ∈ N such that φ\_e = φ\_(h(e)), where φ\_e denotes the e-th computable function.

We will leverage this theorem to prove the existence of an x ∈ N satisfying φ\_x(y) = x^y for all y ∈ N.

Define the function g : N^2 → N as follows:

g(x,y) = x^y

Clearly, g is a computable function. By the s-m-n theorem, there exists a total computable function s : N → N such that for all x,y ∈ N:

φ\_(s(x))(y) = g(x,y) = x^y

Now, apply the Second Recursion Theorem to the function s. This guarantees the existence of an index e ∈ N satisfying:

φ\_e = φ\_(s(e))

Consequently, for all y ∈ N:

φ\_e(y) = φ\_(s(e))(y) = g(e,y) = e^y

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Descrizione generata automaticamenteThus, by setting x = e, we have successfully constructed an x ∈ N with the property that φ\_x(y) = x^y for all y ∈ N, as desired.

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